Homogeneous Function:-

An expression of the type

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n,$$

in which every term is of degree n, is called a homogeneous function of degree n in x and y.

or $x^n\left[a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \cdots + a_n\frac{y^n}{x^n}\right]$

$$= x^n f(y/x) \quad \text{(say)}$$

is also a homogeneous function of degree n in x and y.

Euler's theorem on Homogeneous functions:-

If u is a homogeneous function of degree n in x & y; then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof:- Let us consider $u = x^n f\left(\frac{y}{x}\right)$, a homogeneous function of degree n. Then

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right)\cdot nx^{n-1}$$

(treating y as constant)

$$\Rightarrow x\frac{\partial u}{\partial x} = -yx^{n-1} f'\left(\frac{y}{x}\right) + nx^n f(y/x) \quad \text{(i)}$$

Again; $\frac{\partial u}{\partial y} = x^n. f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$

$$\therefore y\frac{\partial u}{\partial y} = x^{n-1}y. f'(y/x) \quad \text{(ii)}$$

Adding (i) & (ii); we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(y/x) = nu.$

This theorem is hold good also in case if u is a homogeneous function of degree n in a finite number of variables $x, y, z, \ldots, w$.

Then the function $x^n f\left(\frac{y}{x}, \frac{z}{x}, \ldots, \frac{w}{x}\right)$ and the theorem gives the result

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \cdots + w\frac{\partial u}{\partial w} = nu.$$

Corrollary: If $f(x,y)$ is a homogeneous function of degree n, then we have shown that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ which can be put in the form

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f = nf \quad \text{—— (I)}$$

i.e. when the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is applied to the function f, then produces a function which is just the function f multiplied by n.

If the operator applied to (I) again, we get

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = n\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\}$$

$$\Rightarrow x\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = n.nf$$

$$\Rightarrow x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x\partial y} + xy\frac{\partial^2 f}{\partial y\partial x} + y^2\frac{\partial^2 f}{\partial y^2} + y\frac{\partial f}{\partial y} = n^2 f$$

$$\Rightarrow x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n^2 f - nf = n(n-1)f \quad \text{—— (II)}$$

Similarly, if the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is applied (II), we get

$$x^3\frac{\partial^3 f}{\partial x^3} + 3x^2y\frac{\partial^3 f}{\partial x^2\partial y} + 3xy^2\frac{\partial^3 f}{\partial x\partial y^2} + y^3\frac{\partial^3 f}{\partial y^3} = n(n-1)(n-2)f \quad \text{—— (III)}$$

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The general form can be expressed as

$$x^m \frac{\partial^m f}{\partial x^m} + {}^mC_1 x^{m-1} y \cdot \frac{\partial^m f}{\partial x^{m-1}\partial y} + {}^mC_2 x^{m-2} \cdot y^2 \frac{\partial^m f}{\partial x^{m-2}\partial y^2} +$$

$$\cdots + {}^mC_r x^{m-r} y^r \frac{\partial^m f}{\partial x^{m-r}\partial y^r} + \cdots + y^m \frac{\partial^m f}{\partial y^m}$$

$$= n(n-1)(n-2)\cdots\cdots(n-m+1) f \quad \text{--- (1)}$$

Example (1): — Verify Euler's theorem when

$$u = \frac{x(x^3-y^3)}{x^3+y^3}$$

Solution: — The degree of function is one.

So; we have to show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

Now; $\log u = \log x + \log(x^3-y^3) - \log(x^3+y^3)$

$$\therefore \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3-y^3} - \frac{3x^2}{x^3+y^3}$$

$$\& \frac{1}{u}\frac{\partial u}{\partial y} = -\frac{3y^2}{x^3-y^3} - \frac{3y^2}{x^3+y^3}$$

$$\Rightarrow \frac{x}{u}\frac{\partial u}{\partial x} = 1 + \frac{3x^3}{x^3-y^3} - \frac{3x^3}{x^3+y^3}$$

$$\& \frac{y}{u}\frac{\partial u}{\partial y} = -\frac{3y^3}{x^3-y^3} - \frac{3y^3}{x^3+y^3}$$

Adding them

$$\frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 1 + \frac{3(x^3-y^3)}{x^3-y^3} - \frac{3(x^3+y^3)}{x^3+y^3} = 1+3-3$$

$$= 1$$

$$\therefore x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Hence the theorem is verified.

Example ② :— If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$; show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

Solution :— It is given that $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

So $\sin u = \frac{x^2+y^2}{x+y} = f(x,y)$ (say)

From above, we find that $f(x,y)$ is a homogeneous function of degree 1. Therefore, by Euler's theorem we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u.$$ (Hence proved)

Exercise :— ① Verify Euler's theorem when $u = x^3+y^3+3xyz$

Exercise ② :— If $u = \sin^{-1}(x/y) + \tan^{-1}(x/y)$; show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Exercise ③ :— If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, show by Euler's theorem that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

To change the single independent variable into the dependent variable :- In $y = f(x)$, let y be dependent variable, where x is independent variable; now we want to change the independent variable into dependent variable i.e

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \left(\frac{dx}{dy}\right)^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dx}{dy}\right]^{-1} = -\left(\frac{dx}{dy}\right)^{-2} \cdot \frac{d^2x}{dy^2} \cdot \frac{dy}{dx} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{\left(\frac{d^2x}{dy^2}\right)}{\left(\frac{dx}{dy}\right)^3}\right] = \frac{\frac{d^3x}{dy^3} \cdot \frac{dy}{dx} \cdot \left(\frac{dx}{dy}\right)^3 - 3\left(\frac{dx}{dy}\right)^2 \left(\frac{d^2x}{dy^2}\right)^2 \cdot \frac{dy}{dx}}{\left(\frac{dx}{dy}\right)^6}$$

$$= \left[\frac{d^3x}{dy^3} \cdot \frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2\right] \Big/ \left(\frac{dx}{dy}\right)^5$$

Example :- Change independent variable x into y when

$$\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$$

Solution :- Here $\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$

$$\rho = \left\{\left[1 + \left(\frac{dx}{dy}\right)^{-2}\right]\right\}^{3/2} \Big/ \left(\frac{d^2x}{dy^2}\right)^{-1}$$

$$= \left\{1 + \left(\frac{dx}{dy}\right)^{-2}\right\}^{3/2} \Big/ \left\{-\left(\frac{dx}{dy}\right)^{3} \cdot \frac{d^2x}{dy^2}\right\}$$

$$= \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{3/2} \Big/ -\left(\frac{dx}{dy}\right)^3 \cdot \left(\frac{dx}{dy}\right)^{-3} \cdot \frac{d^2x}{dy^2}$$

$$\rho = \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{3/2} \Big/ \left\{-\frac{d^2x}{dy^2}\right\}$$ which is required.

To change the independent variable x into another variable t, where $x = f(t)$:

We know; $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}$

The operator $\frac{d}{dx}$ is therefore equivalent to the operator $\frac{1}{\left(\frac{dx}{dt}\right)} \cdot \frac{d}{dt}$.

Therefore $\frac{d^2y}{dx^2} = \frac{1}{\left(\frac{dx}{dt}\right)} \cdot \frac{1}{dt}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]$

$$= \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)^2} = \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

and $\frac{d^3y}{dx^3} = \frac{1}{\left(\frac{dx}{dt}\right)} \cdot \frac{d}{dt}\left[\frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}\right]$

$$= \frac{1}{\left(\frac{dx}{dt}\right)}\left[\left\{\left(\frac{d^3y}{dt^3}\cdot\frac{dx}{dt} + \frac{d^2x}{dt^2}\cdot\frac{d^2y}{dt^2}\right) - \left(\frac{d^3x}{dt^3}\cdot\frac{dy}{dt} + \frac{d^2x}{dt^2}\cdot\frac{d^2y}{dt^2}\right)\right\}\left(\frac{dx}{dt}\right)^3 - 3\left(\frac{dx}{dt}\right)^2\frac{d^2x}{dt^2}\left\{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right\}\left(\frac{dx}{dt}\right)^{-6}\right]$$

$$\frac{d^3y}{dx^3} = \frac{\frac{d^3y}{dt^3}\left(\frac{dx}{dt}\right)^2 - 3\frac{d^2x}{dt^2}\cdot\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} + 3\left(\frac{d^2x}{dt^2}\right)^2\frac{dy}{dt} - \frac{dx}{dt}\cdot\frac{dy}{dt}\cdot\frac{d^3x}{dt^3}}{\left(\frac{dx}{dt}\right)^5}$$

and similar other higher order differential coefficients can be written.

Example: Change the independent variable from x to θ in the equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Having given $x = \cos\theta$

Solution: We have, $x = \cos\theta$

then $\frac{dx}{d\theta} = -\sin\theta$; $\frac{d^2x}{d\theta^2} = -\cos\theta = -x$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{1}{\sin\theta}\cdot\frac{dy}{d\theta}$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\left(\frac{dx}{d\theta}\right)}\cdot\frac{d}{d\theta}\left\{-\frac{1}{\sin\theta}\cdot\frac{dy}{d\theta}\right\}$$

$$= -\frac{1}{\sin\theta}\left\{\frac{-\sin\theta\cdot\frac{d^2y}{d\theta^2} + \cos\theta\cdot\frac{dy}{d\theta}}{\sin^2\theta}\right\}$$

$$= \frac{1}{\sin^3\theta}\cdot\left\{\sin\theta\cdot\frac{d^2y}{d\theta^2} - \cos\theta\cdot\frac{dy}{d\theta}\right\},$$

Putting these values in given eqn becomes,

$$(1-\cos^2\theta)\left\{\frac{\sin\theta\cdot\frac{d^2y}{d\theta^2} - \cos\theta\frac{dy}{d\theta}}{\sin^3\theta}\right\} + \frac{\cos\theta}{\sin\theta}\cdot\frac{dy}{d\theta} + y = 0$$

$$\Rightarrow \frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\cdot\frac{dy}{d\theta} + \frac{\cos\theta}{\sin\theta}\cdot\frac{dy}{d\theta} + y = 0$$

$\Rightarrow \frac{d^2y}{d\theta^2} + y = 0$ which is required eqn.

Exercise: Transform the differential eqn

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx} - \sin x - 2y\cos^3 x = 2\cos^5 x$$

into one having z as independent variables, where $z = \sin x$.

Example: If $x^2+z^2=1$, show that the equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0$$

becomes $z(z^2-1)\frac{d^2y}{dz^2}+(2z^2-1)\frac{dy}{dz}-n(n+1)zy=0.$

Solution: Here $x^2+z^2=1$ so that $2x+2z\frac{dz}{dx}=0$

$$\Rightarrow \frac{dz}{dx}=-\frac{x}{z}$$

Now; $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{x}{z}\cdot\frac{dy}{dz}$

and $\frac{d^2y}{dx^2}=-\frac{x}{z}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}-\left(\frac{1}{z}-\frac{x}{z^2}\cdot\frac{dz}{dx}\right)\frac{dy}{dz}$

$$=\frac{x^2}{z^2}\frac{d^2y}{dz^2}-\left(\frac{1}{z}+\frac{x^2}{z^3}\right)\frac{dy}{dz}$$

Hence the given eqn becomes;

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

$$\Rightarrow (1-x^2)\left[\frac{x^2}{z^2}\cdot\frac{d^2y}{dz^2}-\frac{x^2+z^2}{z^3}\cdot\frac{dy}{dz}\right]-2x\left(-\frac{x}{z}\cdot\frac{dy}{dz}\right)+n(n+1)y=0$$

$$\Rightarrow z^2\left[\frac{1-z^2}{z^2}\frac{d^2y}{dz^2}-\frac{1}{z^3}\cdot\frac{dy}{dz}\right]+2\frac{(1-z^2)}{z}\frac{dy}{dz}+n(n+1)y=0$$

$$\Rightarrow z(1-z^2)\frac{d^2y}{dz^2}+(1-2z^2)\frac{dy}{dz}+n(n+1)y=0$$

$$\Rightarrow z(z^2-1)\frac{d^2y}{dz^2}+(2z^2-1)\frac{dy}{dz}-n(n+1)y=0.$$

(Hence proved).

Exercise: Show that the equation $\frac{d^2x}{dy^2}=a$ may be written in the form $\frac{d^2y}{dx^2}+a\left(\frac{dy}{dx}\right)^3=0$.

Transformation from cartesian to polars and vice-versa:— It often happens that a result in Cartesian is much simplified on reduction to polar or vice-versa.

Let $x = r\cos\theta,\ y = r\sin\theta$

Suppose θ to be independent variable, then

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx}\cdot\frac{d}{d\theta}\left\{\frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cdot\cos\theta - r\sin\theta}\right\}$$

$$= \Bigg[\left(\frac{dr}{d\theta}\cos\theta + \frac{d^2r}{d\theta^2}\sin\theta - r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$$
$$\times\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) - \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)\times$$
$$\left(-\frac{dr}{d\theta}\sin\theta + \frac{d^2r}{d\theta^2}\cos\theta - r\cos\theta - \frac{dr}{d\theta}\sin\theta\right)\Bigg] \div \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^3$$

$$\frac{d^2y}{dx^2} = \Bigg[2\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 3r\frac{dr}{d\theta}\sin\theta\cos\theta + \frac{d^2r}{d\theta^2}\sin\theta\cos\theta\frac{dr}{d\theta}$$
$$- r\frac{d^2r}{d\theta^2}\sin^2\theta + r^2\sin^2\theta + 2\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 3r\frac{dr}{d\theta}\sin\theta\cos\theta$$
$$- \frac{d^2r}{d\theta^2}\sin\theta\cos\theta\frac{dr}{d\theta} - r\frac{d^2r}{d\theta^2}\cos^2\theta + r^2\cos^2\theta\Bigg] \div \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^3$$

$$= \frac{2\left(\frac{dr}{d\theta}\right)^2 + r^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^3} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^3}.$$

Example: Transform $p = \dfrac{x\frac{dy}{dx} - y}{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}}$ into polars.

Solution :- Multiplying numerator & denominator by $\frac{dx}{dt}$, we get

$$p = \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \Big/ \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \frac{r^2\frac{d\theta}{dt}}{\sqrt{\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\}}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}{r^4\left(\frac{d\theta}{dt}\right)^2} = \frac{1}{r^2} + \frac{1}{r^4}\cdot\frac{\left(\frac{dr}{dt}\right)^2}{\left(\frac{d\theta}{dt}\right)^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Example :- ② Transform $\tan\phi = r\frac{d\theta}{dr}$ into Cartesian form.

Solution :- We have $\tan\phi = r\frac{d\theta}{dr} = \frac{r^2\frac{d\theta}{dt}}{r\frac{dr}{dt}}$

$$= \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x\frac{dx}{dt} + y\frac{dy}{dt}}$$

$$\Rightarrow \tan\phi = \frac{\left\{x\frac{dy}{dx} - y\right\}\frac{dx}{dt}}{\left\{x + y\frac{dy}{dx}\right\}\frac{dx}{dt}} = \frac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}$$

Exercise :- Transform the formula $\rho = \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}$ into polar coordinates.

x and y to be expressed in terms of some third variable :- To show that

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \quad \text{--- (I)}$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt} \quad \text{--- (II)}$$

$$\& \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad \text{--- (III)}$$

Proof:

We have

$$x = r\cos\theta \; ; \quad y = r\sin\theta$$

$$\& \quad x^2 + y^2 = r^2$$

Differentiating, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$ this is result Ist.

To prove result (III) we will find the value of $\left(\frac{ds}{dt}\right)^2$ in two forms

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dx}\right)^2 \left(\frac{dx}{dt}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\left(\frac{dx}{dt}\right)^2$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \quad \text{--- (a)}$$

Again;

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{ds}{d\theta}\right)^2 \cdot \left(\frac{d\theta}{dt}\right)^2 = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}\left(\frac{d\theta}{dt}\right)^2$$

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 \quad \text{--- (b)}$$

Equating the two values,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2$$ proved.

To prove 2nd relation, multiplying (iii) by r^2 & subtracting the square of relation (I) from it

$$(x^2+y^2)\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}=r^2\left(\frac{dr}{dt}\right)^2+r^4\left(\frac{d\theta}{dt}\right)^2 \quad \text{--- (c)}$$

squaring Ist relation,

$$x^2\left(\frac{dx}{dt}\right)^2+y^2\left(\frac{dy}{dt}\right)^2+2xy\frac{dx}{dt}\cdot\frac{dy}{dt}=r^2\left(\frac{d\theta}{dt}\right)^2 \quad \text{--- (d)}$$

subtracting (d) from (c), we have

$$y^2\left(\frac{dx}{dt}\right)^2+x^2\left(\frac{dy}{dt}\right)^2-2xy\frac{dx}{dt}\cdot\frac{dy}{dt}=r^4\left(\frac{d\theta}{dt}\right)^2$$

$$\Rightarrow \left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)^2=\left(r^2\frac{d\theta}{dt}\right)^2$$

$$\Rightarrow y\frac{dx}{dt}-x\frac{dy}{dt}=r^2\frac{d\theta}{dt}$$ result IInd proved.

Example :- Transform $\frac{d^2y}{dx^2}$ to the new variables u & v taking u as the independent variable, given $x=v^{-1}$; $y=uv$.

Solution :- From the given relations, we have

$$\frac{dx}{du}=-\frac{1}{v^2}\frac{dv}{du} \quad \& \quad \frac{dy}{du}=v+u\frac{dv}{du}$$

Now, $$\frac{dy}{dx}=\frac{dy/du}{dx/du}=\frac{v+u\frac{dv}{du}}{-\frac{1}{v^2}\frac{dv}{du}}$$

& $$\frac{d^2y}{dx^2}=\frac{1}{\left(\frac{dx}{du}\right)}\cdot\frac{1}{du}\left[\frac{v+u\frac{dv}{du}}{-\frac{1}{v^2}\frac{dv}{du}}\right]=\frac{v^2}{\left(\frac{dv}{du}\right)}\cdot\frac{1}{du}\left\{\frac{v^3+uv^2\frac{dv}{du}}{\frac{dv}{du}}\right\}$$

$$=\frac{v^2}{\left(\frac{dv}{du}\right)^3}\cdot\left[\frac{dv}{du}\left\{3v^2\frac{dv}{du}+v^2\frac{dv}{du}+2uv\left(\frac{dv}{du}\right)^2+uv^2\frac{d^2v}{du^2}\right\}-\left(v^3+uv^2\frac{dv}{du}\right)\cdot\frac{d^2v}{du^2}\right]$$

$$=\frac{v^3}{\left(\frac{dv}{du}\right)^3}\left[4v\left(\frac{dv}{du}\right)^2+2u\left(\frac{dv}{du}\right)^3-v^2\frac{d^2v}{du^2}\right]$$ Required result.